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On the integral relationship for mean angular momentum of gravity waves in finite-depth water

By ZHOUWEN YU AND JIN WU
Air–Sea Interaction Laboratory, College of Marine Studies, University of Delaware, Lewes, DE 19958, USA

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In his recent papers, Longuet-Higgins derived a relation of mean angular momentum for gravity waves in deep water. In this paper, an expression for the mean Eulerian angular momentum in water of arbitrary depth is derived. It differs from Longuet-Higgin's expression by an additive term $2IB/g$ accounting for finite-depth effects, where I is the density of mean horizontal momentum, B the Bernoulli constant and g the gravitational acceleration. In addition, the present derivation appears to be simpler and more straightforward.

1. Introduction

Research on integral properties of progressive, irrotational gravity waves has received considerable attention (Starr 1947*a*, *b*; Longuet-Higgins 1975, 1980, 1984). In his recent papers, Longuet-Higgins (1980, 1984) defined the quantity of angular momentum in progressive gravity waves, and suggested that this quantity is likely to be associated with the phenomenon of wave breaking. Furthermore, adopting a series expansion of space variables (X, y) in terms of velocity potential Φ and stream function Ψ of steady-state waves, he derived a simple relation between the Eulerian mean angular momentum $\bar{A}_{\mathrm{E}}$ and the Lagrangian function $L = T - V$ in water of infinite depth (Longuet-Higgins 1984), i.e.

$$\bar{A}_{\mathrm{E}} = \frac{2c}{g}(T - V), \tag{1}$$

where T and V are densities of mean kinetic and potential energies, c the phase velocity, and g the gravitational acceleration.

In this short note we derived a similar relationship for finite-depth water using a rather different but simpler approach.

2. Derivation

The notations used here are exactly the same as those in Longuet-Higgins (1980, 1984). A train of waves with wavelength λ and phase velocity c is travelling in the x-direction in water of constant depth h. The y-axis is vertically upward, and the origin is chosen at the mean water surface (see figure 1). The pressure at the free surface is assumed to be zero, i.e. $P_0 = 0$. (If $P_0 \neq 0$, we can simply substitute $P - P_0$ for P in the following derivation, and all the expressions still hold.) The density of

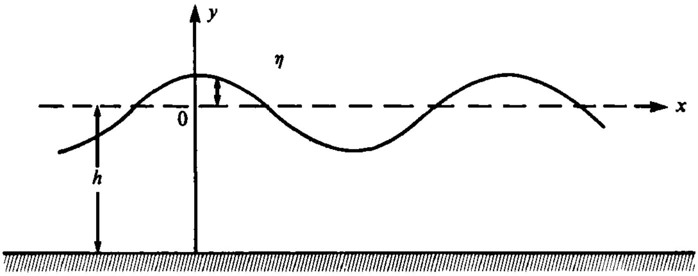

FIGURE 1. A train of progressive waves.

the Eulerian mean angular momentum about the origin is defined as (Longuet-Higgins 1980, 1984)

$$\bar{A}_{\mathrm{E}} = \overline{\frac{1}{\lambda}\int_0^{\lambda}\int_{-h}^{\eta} \rho(yu-xv)\,\mathrm{d}y\,\mathrm{d}x}$$

$$= \overline{\int_{-h}^{\eta} \rho y u\,\mathrm{d}y}$$

$$= \frac{1}{\lambda}\int_0^{\lambda}\int_{-h}^{\eta} \rho y u\,\mathrm{d}y\,\mathrm{d}x, \tag{2}$$

where u and v are components of particle velocity, ρ is the density of water, and the overbar denotes the mean value with respect to time t.

For any point in progressive waves, the following equations hold:

$$\rho\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{\partial p}{\partial x}, \tag{3}$$

$$\rho\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{\partial}{\partial y}(p+\rho g y), \tag{4}$$

$$[\tfrac{1}{2}\rho(u^2+v^2)+p+\rho g y-\rho c u]\,u = \rho B u, \tag{5}$$

where

$$B = \tfrac{1}{2}\overline{u^2_{-h}} = \frac{1}{\lambda}\int_0^{\lambda} \tfrac{1}{2}u^2(x-ct,\,-h)\,\mathrm{d}x, \tag{6}$$

which should be considered to be one of the most important parameters characterizing the wave dynamics in the case of finite depth (Longuet-Higgins 1975).

Multiplying (3) and (4), respectively, by the velocity potential ϕ and the stream function ψ of the progressive waves, and utilizing the following identities:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\partial\psi}{\partial t} = -c\frac{\partial\psi}{\partial x} = cv, \tag{7}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = u^2+v^2-cu, \tag{8}$$

we readily obtain

$$pu+\rho(u^2+v^2)\,u = \rho\frac{\mathrm{d}u\phi}{\mathrm{d}t}+\frac{\partial p\phi}{\partial x}+\rho c u^2, \tag{9}$$

$$pu = \rho\frac{\mathrm{d}v\psi}{\mathrm{d}t}+\frac{\partial}{\partial y}(p+\rho g y)\,\psi-\rho c v^2-\rho g y u. \tag{10}$$

Elimination of the terms $\frac{1}{2}\rho(u^2+v^2)\,u$ and pu on the left-hand side of (5) by using (9) and (10) leads to

$$\rho g y u = \rho c(u^2+v^2) - \frac{\partial}{\partial y}(p+\rho g y)\,\psi + 2\rho B u - f(x-ct,\, y), \tag{11}$$

where

$$f(x-ct,\, y) = \rho\frac{\mathrm{d}}{\mathrm{d}t}(u\phi) + \rho\frac{\mathrm{d}}{\mathrm{d}t}(v\psi) + \frac{\partial}{\partial x}(p\phi). \tag{12}$$

Integrating (11) over a range of one wavelength and using Green's theorem, we easily obtain the relationship between the Eulerian mean angular momentum $\overline{A}_{\mathrm{E}}$ and the Lagrangian function $L = T - V$ in the case of finite-depth water:

$$\overline{A}_{\mathrm{E}} = \frac{2c}{g}(T-V) + \frac{2I}{g}B, \tag{13}$$

where I is the well-known density of mean horizontal momentum defined by

$$I = \overline{\int_{-h}^{\eta} \rho u \,\mathrm{d}y}. \tag{14}$$

In deriving the above relationship, we have taken into account the fact that

$$\psi_\eta = \text{constant} + c\eta \tag{15}$$

and

$$\int_0^{\lambda}\int_{-h}^{\eta} f(x-ct,\, y)\,\mathrm{d}y\,\mathrm{d}x = 0. \tag{16}$$

Equation (16) can be easily proved if we make use of the following conditions: (i) u, v, ϕ, ψ and p are periodic functions with respect to x and t; (ii) $p_0 = 0$; and (iii) $v\,\mathrm{d}x - (u-c)\,\mathrm{d}y = 0$ at the surface (kinematic boundary condition at the free surface).

Note that in the case of infinite depth $B = 0(\lim_{h\to\infty} Bh = 0)$; hence (13) reduces to (1) as derived by Longuet-Higgins.

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